Qualifying Exam Syllabus & Transcript

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Quals Syllabus

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1 Major topic: Algebraic Number Theory (Algebra)

- **Number Fields:** integrality, norm and trace, Dedekind domains, ideal factorization and class group, lattices and Minkowski bound, Dirichlet's unit theorem
- **Local Theory:** p-adic numbers, completions, valuations and absolute values, extensions of valuations, Hensel's lemma, local and global fields, ramification of extensions
- **Class Field Theory:** adeles and ideles, statements of local and global class field theory, statement of Artin reciprocity, statement of Chebotarev density

2 Major topic: Probability Theory (Probability)

- **Preliminaries:** σ-algebras, Dynkin's π-λ theorem, independence, Borel–Cantelli lemmas, Kolmogorov's 0-1 law, Kolmogorov's maximal inequality, strong and weak laws of large numbers
- **Central limit theorems:** weak convergence, characteristic functions, tightness, I.I.D. central limit theorem, Lindeberg–Feller central limit theorem
- **Conditioning:** conditional probability and expectation, regular conditional probabilities
- **Martinagles:** stopping times, upcrossing inequality, uniform integrability, A.S. convergence, Doob's decomposition, Doob's inequality, L^p convergence, L^1 convergence, reverse martingale convergence, optional stopping theorem, Wald's identity
- **Markov chains:** countable state space, stationary measures, convergence theorems, recurrence and transience, asymptotic behavior

3 Minor topic: Complex Analysis (Analysis)

- **Complex functions:** holomorphic, meromorphic, Cauchy-Riemann equations, Liouville's theorem, Taylor and Laurent series
- **Complex integration:** Cauchy's theorem, Cauchy's integral formula, residue theorem, argument principle, Rouché's theorem, Morera's Theorem, maximum modulus principle
- **Fundamental Theorem of Algebra:** Statement and proof

Quals Transcript

Note: This qualifying exam was on Zoom to accomodate a faculty member who was not in Berkeley.

Number Theory

Sug Woo: Have you chosen what order you want to take the topics?

Anya: Yeah, let's start with number theory.

Melanie: Okay, I'll start. What's the class group of $\mathbb{Q}(\sqrt{-5})$?

Anya: $\mathbb{Z}/2\mathbb{Z}$

Melanie: Tell me about how we know that.

I mentioned the Minkowski bound for restricting to particular prime ideals generating the class group and finding relations among them. I give the Minkowski bound $M_K = \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s\sqrt{|d_K|}$ and define the terms in the bound. For this case we get $M_K = \frac{2\sqrt{20}}{\pi} < 3$. Initially I bounded by $\frac{10}{\pi}$ but Jim later pointed out this isn't less than 3, at which point Melanie confirmed that the actual M_K is still less than 3 so this is the bound I used.

Next I related the Minkowski bound to the class group in terms of bounding norms of ideal representatives, and made the reduction to looking at prime ideals by unique factorization. To find prime ideals with norm bounded by 3 we just need to look at primes above (2), which I show is ramified using Dedekind-Kummer. I then point out the order of the prime ideal dividing (2) is 2 in the class group since (2) is principal.

Melanie: How do you know it's not order 1?

Anya: We have to check that there isn't a principal generator for the prime ideal. Let's say that there was then the norm of the generator is the norm of the prime ideal which we know is 2 –

Melanie: Yep, okay great. That was good, I'll let Sug Woo ask a question.

Sug Woo: Consider the cyclotomic extension generated by nth roots of unity over $\mathbb{Q}$. As an extension of $\mathbb{Q}$, tell us about any interesting things you know about the extension. For instance, splitting primes or Galois group or discriminant or...

I state the ring of integers $\mathcal{O}_K$ and the Galois group as $(\mathbb{Z}/n\mathbb{Z})^*$ with a map $\zeta_n \mapsto \zeta_n^m$ getting sent to $m \mod n$. I talk about why these are exactly the maps in the Galois group using the fact that it is a primitive extension and ζ_n must get sent to another *primitive* root of unity.

Sug Woo: Okay, what about rational primes split completely in this extension, can we identify them?

[a brief pause] I'm not sure where to start so I start saying things I can think of that are true.

Anya: Well if it splits completely in $\mathbb{Q}(\zeta_n)$ it splits completely in any subextension, so in any $\mathbb{Q}(\zeta_p)$ if p divides n. Its decomposition group would be trivial in this case.

[another pause, Sug Woo starts to talk just as I do and he tells me to go ahead]

Anya: Going back to the previous problem using Dedekind Kummer, then a prime would split completely exactly when the minimal polynomial, so the nth cyclotomic polynomial, splits completely mod p which gives a partial characterization.

Sug Woo: Another way is if you knew where the Frobenius element in the Galois group goes to in $(\mathbb{Z}/n\mathbb{Z})^*$ using the identification you gave, then you could get the answer but then you need to know where the Frobenius elements go. But you can take any approach you like.

Anya: Right, so if the prime splits completely it has trivial decomposition group so the Frobenius would be the identity automorphism so we want to find the primes with trivial Frobenius.

Sug Woo: Can you make a guess what Frob_p should map to in $(\mathbb{Z}/n\mathbb{Z})^*$?

Anya: I would guess $p \mod n$.

Sug Woo: Yeah that's right, so if you can prove that you'll get the answer.

Anya: Okay, well the Frobenius acts like $x \mapsto x^p$ so in particular it sends $\zeta_n \mapsto \zeta_n^p$ which maps to $p \mod n$ in the map from before. So then we just need to figure out which are trivial which is the primes such that $p \equiv 1 \mod n$.

Sug Woo: There are a few things one can make a bit more precise. Let's make one thing a little more precise, can we characterize the Frobenius elements? We identified Frob_p with p just by looking at the action on ζ_n, but with general situations in mind what is the characterization of the Frobenius element?

Anya: It's a generator for the decomposition group mod the inertia subgroup, and in particular the generator that acts by $x \mapsto x^p$.

Sug Woo: Sounds good, I'm sure you could write down a formula for that. I'm satisfied. Let's turn it over to Melanie.

Melanie: Can you tell me what the cyclic cubic extensions of the rationals are?

I say this looks like Class Field Theory so I write out the global artin map for $\mathbb{Q}$ and describe its relationship to finite abelian extensions of $\mathbb{Q}$. Then I simplify to the $\mathbb{Z}/3\mathbb{Z}$ Galois group on the right. I want to simplify the idele class group expression, so I talk about the map inducing an isomorphism on the profinite completions and also when we quotient $C_{\mathbb{Q}}$ by the connected component of 1. I define the idele class group and mention the topologies on $\mathbb{Q}_p$ are all totally disconnected so the connected component is $(1 \times \cdots \times 1 \times \mathbb{R}^+)/\mathbb{Q}^*$ and state that the resulting quotient is $\prod_p \mathbb{Z}_p^*$.

Anya: Would you like me to go into the proof of the quotient or just use it?

Melanie: Maybe say what facts are special about $\mathbb{Q}$ that you use to prove it. Would this be true for a general K?

Anya: One thing that's nice is that there's only 1 infinte place, so maybe in the finite places the proof would run similarly using uniformizers in each place, but I'm not sure what would happen with more infinite places.

Melanie: Okay, I'm going to push you on the finite places. Suppose we replace $\mathbb{Q}$ with K and we have one uniformizer in one place and I don't have positive valuation in any other place and want to show this is equivalent to something in the product of local units. What do I need to do?

Anya: You want to find something in K^* that cancels exactly that one uniformizer.

Melanie: Do I expect that to exist for a general K?

Anya: Maybe not (I'm still not sure why at this point)

Melanie: Why not?

I guess unique factorization and then talk myself out of it.

Melanie: Let's say we have a K and we want an element that has valuation 1 at this prime and valuation zero at all other primes. Does such an element always exist?

Anya: Maybe you run into an issue when a rational prime splits into multiple primes and there's some connection between the uniformizers?

Melanie: Let's say you have a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and an element that valuation 1 at $\mathfrak{p}$ and valuation zero everywhere else. What can you say about the relationship between the element and the ideal $\mathfrak{p}$?

Anya: Taking the prime ideal factorization of the ideal generated by the element you would just have $\mathfrak{p}$, and no other primes.

Melanie: Do you expect that to happen for every prime of $\mathcal{O}_K$?

Anya: So this is the same thing as saying every prime ideal is principal, which is not always true as we saw in the first example.

Melanie: Right, but it's true for $\mathbb{Q}$ so you were implicitly using that for the formula. Sorry for the sidetrack, so let's say you have the product $\prod_p \mathbb{Z}_p^*$.

I plug this back into the CFT expression from before so we want to enumerate surjective maps $\prod_p \mathbb{Z}_p^* \to \mathbb{Z}/3\mathbb{Z}$. Given the structure on the right, we know cubes are in the kernel, so we want to find $(\mathbb{Z}_p^*)^3$ for all p. I start writing out the structure for $\mathbb{Z}_p^*$ for p even and odd and start to work through what the cubes are and the resulting quotients. I get through $p = 2$ and then explain why $p = 3$ is specia and do this case also.

Melanie: I'm going to interupt you because I think you can do this and you have a systematic plan for writing this out as an additive group and then find the homomorphisms to $\mathbb{Z}/3\mathbb{Z}$. So let's say you did this and you've got a cyclic cubic extension, and I want to know what its discriminant is. Could you tell me in terms of this set up?

I talk about how to determine ramification for each prime from the image of $\mathbb{Z}_p^*$ under the artin map to the inertia subgroup and how this gives the prime divisors for the discriminant but I'm not sure how to get it more explicitly.

Melanie: Do you know anything else about how you might find the discriminant of a number field if you know the primes and how they ramify?

Anya: Well the primes ramify if and only if they divide the discriminant but there could still be powers happening in the discriminant.

Melanie: Alright, I'm good with that question.

Sug Woo: Okay maybe let's ask something about local fields. Let's take $\mathbb{Q}_p$ and attach ζ_q for q a prime (possibly the same prime). Can we tell if this is ramified and if it is ramified can we tell how it is ramified (e.g. totally ramified)?

I mention Dedekind Kummer again to get prime splitting in terms of the splitting of $\Phi_q(x)$.

Sug Woo: Before we compute anything do you have any guesses for when it will be ramified?

I'm really not sure, I hazard a guess but then say I'd rather work through it because I don't know.

I mention local class field theory and set up the local artin map for this case. I then expand the structure for $\mathbb{Q}_p^*$.

Sug Woo: If we do this it might be helpful to identify the Galois group of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_q)/\mathbb{Q}_p)$.

I start with the global galois group again which would be $(\mathbb{Z}/q\mathbb{Z})^*$.

Sug Woo: That's going to be a little tricky...

Anya: Well it comes back to the question of splitting for Φ_q over $\mathbb{Q}_p$.

Sug Woo: One potential problem is that $\mathbb{Q}_p$ can contains some extra roots of unity. Do you what extra roots of unity $\mathbb{Q}_p$ contains exactly?

Anya: Well I know you can figure it out using the decomposition of $\mathbb{Z}_p^* = \mathbb{Z}/p-1\mathbb{Z} \times \mathbb{Z}_p$ so $\zeta_\ell \in \mathbb{Q}_p$ if and only if $\ell \mid p-1$.

Sug Woo: So if you adjoin ζ_ℓ satisfying this condition your extension field is trivial. I'll suggest an approach. You can compute the discriminant of the cyclotomic extension in the local case, or in the global case $\mathbb{Q}(\zeta_q)/\mathbb{Q}$ and check ramification there. You could also use Eisenstein stuff or $1 - \zeta_q$. And you don't need the discriminant precisely just the prime factors.

Anya: Well I don't know it precisely but I do know the prime factors. In this case $\mathrm{Disc}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ is q to some power. [I also mention what it would be for ζ_n] So knowing that this is our discriminant, the only the ramified prime in the global extension is (q). So if $p \neq q$ then we are unramified in the local extension and if $p = q$ then we are ramified.

Sug Woo: Can you say whether it is totally ramified in the case $p = q$? Again, there is more than one way to see it.

Anya: Well if it's totally ramified, the inertia subgroup is the whole Galois group which might give one way of testing that... I guess you could again look at the minimal polynomial and see if that's totally ramified mod p.

Sug Woo: One of the easier ways is based on the minimal polynomial, can you apply the Eisenstein criterion?

Anya: Well we want to look at the minimal polynomial $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$.

Sug Woo: This may not be minimal a priori, but it is a polynomial with ζ_p as a root.

I talk about showing this is irreducible globally by writing it as $\frac{x^p-1}{x-1}$ and switching to $x = y + 1$, which preserves irreducibility. I plug in and cancel to get $y^p + py(\bullet) + p$ and then apply Eisenstein's to get that this is irreducible. I then argue this holds not just globally but also for $\mathbb{Q}_p$ since (p) is still a prime ideal.

Then returning to the ramification question, (p) is totally ramified if and only if $\Phi_p(x)$ is totally ramified mod p, so we want $1 + x + \cdots x^{p-1} = (\bullet)^{p-1}$. I then switch to writing it as $\frac{x^p-1}{x-1}$ and then using mod p rewrite as $\frac{(x-1)^p}{x-1} = (x-1)^{p-1}$ (I worry about $p = 2$ for a moment and then explain why that's okay). Then this is totally ramified mod p so (p) is totally ramfied in $\mathbb{Q}_p(\zeta_p)$.

[Sug Woo and Melanie decide they are happy with number theory and we take a few minute break.]

Probability

Jim: There's a rather quick way to prove the strong law in the iid case if you assume a few extra moments, do you know that? Could you talk about it?

Anya: Yeah, specifically the fourth moment?

Jim: Yes exactly. The proof combines some of the concepts in the preliminaries, could you walk us through that?

I state the IID Strong Law of Large Numbers and add the assumption that $EX_i^4 < \infty$ to simplify the proof. First I apply Chebyshev's Inequality to $|S_n/n|$, Jim points out that I made a reduction and I clarify that we recenter to $\mu = 0$ first. I then use Chebyshev to bound in terms of ES_n^4, and then a combinatorial argument and Jensen's inequality to bound this by a constant times $n^2 EX_i^4$.

Jim: Okay I'll give you constant over n^2 at the end of this. [I write C/n^2] How does that help you?

I then state and apply the Borel-Cantelli Lemma to conclude that $P(|S_n/n| > \varepsilon \text{ infinitely often}) = 0$ so that $S_n/n \to 0 = \mu$ a.s. [The details of this proof can be found in Durrett, Theorem 2.3.5]

Jim: Couldn't have done that better myself, very clear. Anyone else have questions about this? [nope]

Jim: Moving on to Central Limit Theorem, you've got some discussion of characteristic functions and tightness, maybe you could remind us what the characteristic function is for a random variable X.

I define $\varphi_X(t)$ in general and give a nice formula in the case that X has a density function $f(x)$.

Jim: Right, that gives us a nice formula to get φ from f. Suppose you had a formula for φ, would it be possible to tell that it came from a density like that and if so what density it was?

Anya: This gets into inversion formulas and there's a particularly nice one that describes the case you were hinting at. If φ is integrable, then there is a density $f(x) = \frac{1}{2\pi} \int e^{-ity} \varphi(t) dt$. [I forgot the $\frac{1}{2\pi}$ initially and we latter noticed and came back to add it].

Jim: That's true, and moreover we learn something about f, it's not just any old density, for example you couldn't get the uniform distribution on $(0, 1)$ like this.

Anya: Well it's absolutely continuous because it has a density.

Jim: Well yes but more than that. If you start off with uniform on $(0, 1)$ you would find that its characteristic function is not integrable and so you couldn't apply this formula. If you added to uniforms then you'd get something triangular and the characteristic function would be $\sin^2(t)/t^2$ and you'd be in better shape. So with your assumption that φ is integrable do you learn more about f than it just being a density? Look at your formula, what happens if you change your y in the formula?

[I'm really not sure where he's going with this yet]

Anya: Well we have e^{-ity} which doesn't change the magnitude and the $\varphi(t)$ is integrable...

Jim: Yes, keep going...

[I fumble around for a little bit here and write $\int |e^{-ity}\varphi(t)dt] \leq \int |\varphi(t)|dt]$

Jim: What you've written looks like an absolute bound on the f.

Anya: Okay, so f is a bounded density.

Jim: Right. Push a little further?

Fraydoun chimes in that Jim is getting at some kind of regularity.

Jim: Well you've got a formula with an integral in it and a parameter that varies. Much of the point of probability and measure theory is discussing integrals with parameters that vary. We know a few theorems about them.

I say something about applying bounded convergence theorem since we have a function that's bounded.

Jim: Well the uniform distribution has a bounded density, but it doesn't fall into this framework because it fails the integrability condition. But if you convolve uniform with itself then what is the shape of that? Can you sketch it and the uniform $(0, 1)$ density.

I sketch both uniform $(0, 1)$ and the sum of that with itself.

Jim: Good. This is now an example that fits the conditions of the inversion theorem (possibly up to some recentering). So in what way is this nicer than just the uniform by itself?

Anya: I'm not sure.

Fraydoun comments again on the importance of regularity, but I'm not sure what he means by "regularity" so this doesn't help me. Jim makes some comment about straying over the boundary between analysis and probability.

Jim: In the background we have some theorems of integration that allow us to switch limits and integrals, could you write down one of these theorems (that might be relevant here)?

I list off bounded, monotone, and dominated dominated convergence.

Jim: Dominated seems quite relevant here. [we discuss some of what was written above I'm still lost]. Maybe just write down the dominated convergence theorem.

I write it in terms of $f_n \to f$. I decide the integrable function of the theorem should be φ [Jim confirms]

Jim: Alright, and what question would lead you to a discussion of a sequence of f_ns, the notation is not great with f_n... you've got a continuous input y, what happens inside the integral as you vary the y?

I mention that it's a rotation... more hinting happens that I don't understand.

Jim: Well the property is something to do with limits because we are discussing the dominated convergence theorem. What properties of functions have to do with limits?

Anya: Continuous or differentiable.

Jim: Okay, start with being continuous and look at your formula again.

It finally clicks and I talk through how to apply dominated convergence gives that if $y_n \to y$ then $f(y_n) \to f(y)$ so f is continuous. Then I revisit the uniform distribution plots with one having continuous density and the other not.

Jim: Suppose you have a $\varphi_n(t) = E(e^{itX_n})$ [asks me to write this down explicitly] and suppose $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$. What then? Can you make any next steps? Perhaps you need some more assumptions. What you tell me about the X_n's?

I comment on $\varphi(0)$ and then that continuity of $\varphi(t)$ at $t = 0$ gives tightness on the X_n's which allows us to conclude that X_n converge weakly to some X with characteristic function φ.

Jim asks about the name of the theorem trying to get at the author, but in Durrett it's only labeled as the "Continuity Theorem" so he tells me that it's often called Lévy's Continuity Theorem and we move on.

Fraydoun: [Looks at syllabus] You have something about Martingales and some inequalities associated with them. Could you define martingales and what types of conditions would garauntee convergence?

I write down the definition for martingales.

Fraydoun: Before you go on, what happens if you have a stopping time? First, what is a stopping time. [I define it]

Fraydoun: Let's say you evaluate your martingale at this stopping time N, is there something you can say about it?

Anya: Well, simplifying first if you take a minimum between the stopping time and the index, $X_{N \wedge n}$ this is also martingale.

Jim: What happens if you didn't do the trick with the $N \wedge n$?

I talk about optional stopping to get $EX_N = EX_0$ and mention that this doesn't always hold.

Jim: Do you have a simple example that exhibits that?

I describe a simple random walk on the integers starting at 1 with stopping time N for the first time it hits 0, so that $EX_N = 0$ but $EX_0 = 1$.

Jim: How do you know that N is finite with probability 1? Otherwise the expression X_N is troublesome if $N = \infty$.

I describe breaking it up into intervals where you can't take n steps to the left when $X_i = n$ and bounding these probabilities and looking at the sum to bound the expectation.

Jim: Sure you can do it by solving equations, but there's also a cute way of arguing by a martingale property. Take your $X_{N \wedge n}$ starting at 1 which is nonnegative and has integer values. Is there anything you can tell us?

I state that it converges a.s. and that integer valued implies the convergence must be to 0.

Jim: Great, so there is this martingale convergence theorem you said which we can come back to in a bit, says this converges to 0 and can't get stuck somewhere else which is why N is finite with probability 1. Let's change notation and say we have $M_n \geq 0$ martingale, then the probability that M_n converges in 1. But we can't expect it to converge in L^1 by your previous example. Why is it true that a nonnegative martingale must be converging almost surely? What method do you use to show that?

I mention a.s. convergence theorem and taking $-M_n$ which is bounded above to get finite $\sup_n EX_n^+$.

Jim: Okay so why is the more general theorem true then?

I write out the a.s. convergence theorem and say that the proof uses the upcrossing inequality.

Jim: Great, I just wanted to see that you had the main idea. So you use the upcrossing inequality, taking an interval (a, b), what do you show about the martingale with respect to that interval?

Anya: You show that it only crosses that interval a finite number of times.

Jim: Good, so that tells you that the sequence must be converging to a limit, but you might have ∞ as a limit. A sequence that runs off to $+\infty$ would cross every interval only finitely often.

Anya: Right but that violates our bound on the $\sup_n EX_n^+$.

[some chatter between Jim and Fraydoun]

Fraydoun: Could you state an inequality involving the supremum?

Jim: How about the probability that the supremum is bigger than something?

I start to write down something and get a couple corrections (no expectation on the inside, take sup over n up to some N) at which point I recognize that they are looking for Doob's Inequality and write that down. Then I tie this back to the conversation of convergence which prevents the limit going off to $+\infty$.

Jim: Right, you can also do it with Fatou's lemma, which is often how it's done in textbooks.

Anya: Yeah that's how I've seen it proved.

[Jim and Fraydoun decide they are happy with probability and we take another short break]

Complex Analysis

Melanie asked about the proof of the Fundamental Theorem of Algebra as a lead in to some complex analysis theorems. I stated the theorem and gave a proof using Liouville's Theorem.

Melanie: Big picture, what goes into the proof of Liouville's theorem?

Anya: There's a chain of proofs/theorems. Liouville's theorem you get from the Cauchy Inequalities, which you get by bounding the Cauchy Integral formula for higher derivatives, which comes from Cauchy's Integral Formula which comes from Cauchy's Theorem.

Fraydoun: Can you state Cauchy's Formula and sketch the proof?

I state and prove the theorem in terms of integrating around C. At some point Fraydoun asks if C is a circle, and I say yes for simplicity but it holds for other contours as well then continue. Most of the way through the proof Fraydoun and Jim interupt and say they are happy and convinced that I know the proof.

Fraydoun: How about the fact that you used that the integral of the holomorphic function is zero. Can you say a little about the proof of that?

I then state and prove Cauchy's Theorem using Green's Theorem. First I split f into its real and imaginary component, $f = u + iv$ and $dz = dx + idy$, then multiply out the integral and apply Green's Theorem to get integrals in terms of the partial derivatives of u and v. Once I have written these integrals I get interupted.

Fraydoun: And to show that these integrals are 0 you use a formula.

Anya: Yes, the Cauchy-Riemann Equations.

Fraydoun: Can you say a little bit about why Cauchy-Riemann is true? What is the idea?

I write out the Cauchy-Equations and talk about how different choices of $h \to 0$ in the limit definition of $f'(z)$ give different expressions of the partial derivates and equating the real and imaginary parts yields the Cauchy-Riemann equations.

Fraydoun: Alright, instead of writing out the details suppose I write f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and we write the matrix of all the derivatives it would have a very special form when the function is analytic. Can you tell me what that form is?

I write out $f(x, y) = u(x, y) + iv(x, y)$ and the matrix of derivatives below, then substitute using the Cauchy-Riemann Equations to get

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Fraydoun: Intuitively can you tell us why the derivative should look like this?

I'm not sure what he's looking for so I start talking about complex integration and the ability to choose $h \to 0$.

Fraydoun: Yes, the proof goes in that direction but let me put it differently. Suppose you take a matrix valued function by matrix multiplication. When is this linear function analytic?

Anya: Based on what we have above it should be of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Fraydoun: Right, there is a reason. Complex multiplcation is the same as multiplying by this matrix where $z = a + ib$.

Anya: Right, gotcha.

Jim: Another way to say it is that the map is shifting and rotating.

Sug Woo: I'm an algebraist, so from my perspective this is how you embed $\mathbb{C}$ into $M^2(\mathbb{R})$.

Everyone agrees they are happy with complex analysis. They go off into a breakout room to talk for a few minutes and come back to congratulate me.