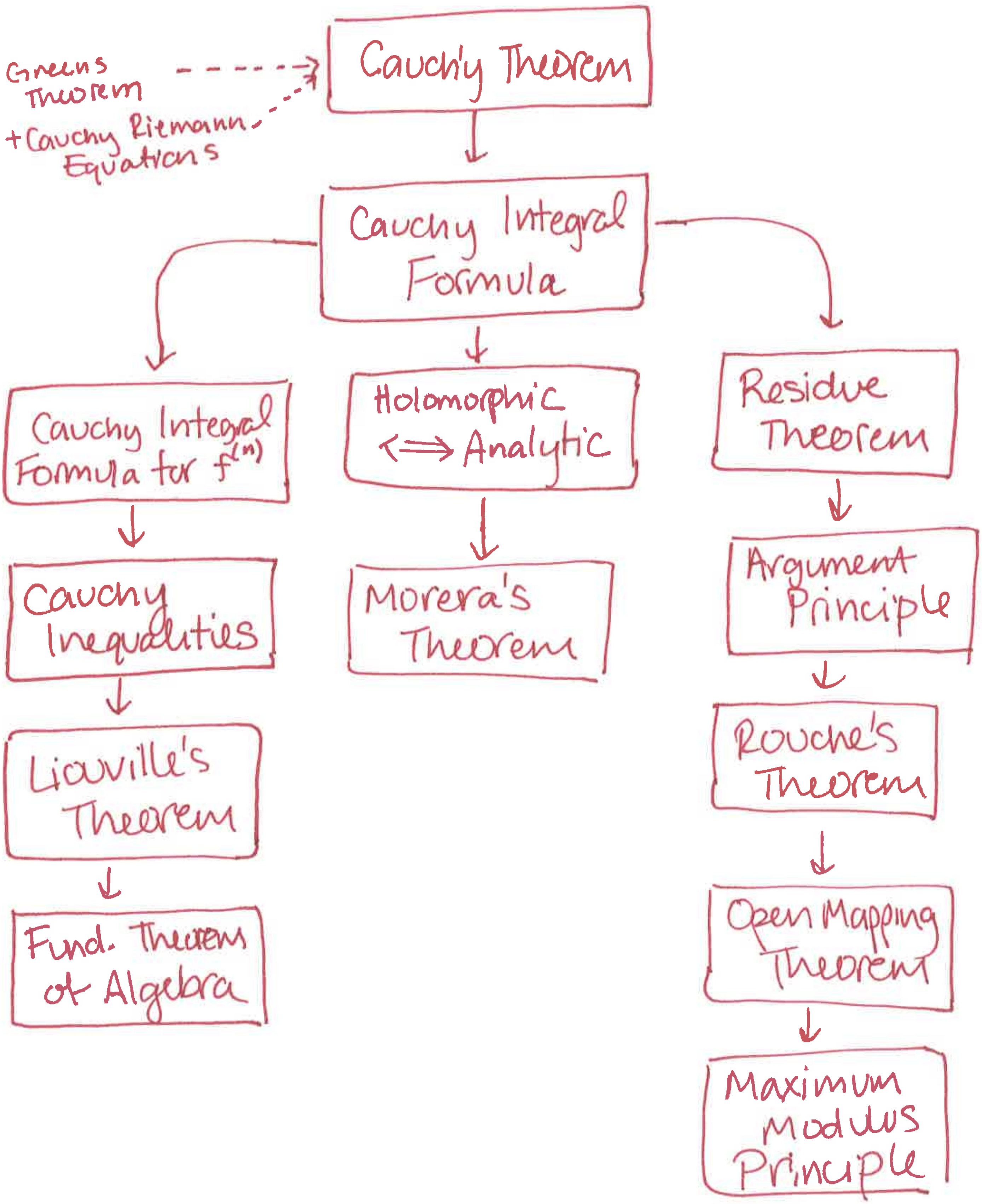
Complex Analysis

Complex Functions - holomorphic, meromorphic, Cauchy-Riemann Equations, Liouville's Theorem Taylor and Laurent Series

complex integration - Cauchy's Theorem, Cauchy's integral formula, residue theorem, argument principle, Rouche's theorem, Morera's Theorem, maximum modulus principle

Fundamental Theorem of Algebra - statement and proof.

Implications



Theorems & Proof Ideas

Cauchy-Riemann Equations:

f holomorphic $\Longrightarrow f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof Idea:

take $h = r \in \mathbb{R}$ and ir as $r \to 0$ in defn of $f'(z)$ and equate real & imag parts.

Cauchy's Theorem

f holomorphic in $\Omega > \gamma$

$$\int_\gamma f(z)\,dz = 0$$

Proof Idea:

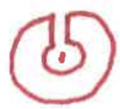
split $f(z)$ & dz in real & imag (w/ or without param first) apply Greens thm $\to$ Cauchy Riem $\to 0$.

Cauchy Integral Formula:

f holomorphic in D bdry C

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$

Proof Idea:



Apply Cauchy to keyhole contour $= 0$. corridors $\to 0$ to get

$$\int_C \frac{f(w)}{w-z} = \int_{C_\varepsilon} \frac{f(w)}{w-z} \rightsquigarrow \underbrace{\frac{f(w)-f(z)}{w-z}}_{\to 0 \text{ by bdd of } f'.} + \underbrace{\frac{f(z)}{w-z}}_{= 2\pi i f(z)}$$

CIF Higher Derivatives:

f holomorphic in D bdry C

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw$$

Proof Idea:

Induction of n start w/ CIF. Take lim derivative for $f^{(n)}(z)$ clever rewrite of $\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}$.

Cauchy Inequalities:

f holomorphic on $\{z : |z - z_0| \leq R\}$

$$|f^{(n)}(z_0)| \leq \frac{n!\,\|f\|_{C_R \text{ bdry}}}{R^n}$$

$$\|f\|_C = \sup_{z \in C} |f(z)|$$

Proof Idea:

Take Cauchy Int. Form for $f^{(n)}(z_0)$ parametrize by $z_0 + Re^{i\theta}$ and bound by $\|f\|_C$.

Theorems & Proof Ideas

Liouville's Theorem:

f entire + bounded
$\Rightarrow$ f constant

Proof Idea:

Cauchy Inequalities bound $|f'| \leq \frac{\|f\|}{R} \to 0$
so $f' = 0$ implies f constant

Fundamental Thm of Algebra:

$P(z)$ nonconstant polynomial $\in \mathbb{C}[z]$
$\Rightarrow P(z)$ has a root in $\mathbb{C}$

Proof Idea:

contradiction: P no roots $\to 1/P$ entire
bdd by $|z| > R$ and $|z| \leq R$ (limits)
apply Liouville $\to 1/P$ constant.

Analytic $\Rightarrow$ Holomorphic:

$f(z) = \sum a_n (z - z_0)^n$
$\Rightarrow f'(z)$ exists
(infinitely differentiable)

Proof Idea:

Write derivate as limit, switch w/ sum
by uniform conv. & take der term by term.
Hadamard gives same radius of conv.

Holomorphic $\Rightarrow$ Analytic:

$f'(z)$ exists
$\Rightarrow f(z) = \sum a_n (z - z_0)^n$ near z_0

Proof Idea:

Cauchy Integral Formula $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw$
Expand $\frac{1}{w - z} = \frac{1}{w - z_0} \sum \left(\frac{z - z_0}{w - z_0}\right)^n$ switch $\int \sum$
and apply CIF for $f^{(n)}(z_0)$ to get expansion.

Morera's Theorem:

f cont. on D, $\forall \Delta \subset D$
$\int_\Delta f(z)\, dz = 0 \Rightarrow f$ holomorphic

Proof Idea:

construct $F(z) = \int_\gamma f(w)\, dw \quad \gamma : z_0 \to z$
show $F'(z) = f(z)$, F holo $\Rightarrow$ analytic
so ∞ diff so f holomorphic too.

Theorems & Proof Ideas

Residue Formula:

f holomorphic except poles $z_1, \dots, z_N$ inside C

$\Rightarrow \int_C f(z)\,dz = 2\pi i \sum \mathrm{res}_{z_i}(f)$

Proof Idea:

keyhole contour + Cauchy Thm to break up $\int_C = \sum \int_{C_i}$.

expand f for each z_i to compute $\int_{C_i}$.

Argument Principle:

f meromorphic, no poles/zeros on C

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = (\#\text{ of zeros in } C) - (\#\text{ of poles in } C)$$

Proof Idea:

use expansion of f to find poles/residues of f'/f. Apply Residue formula.

Rouche's Theorem:

f, g holomorphic on C & int.

$|f(z)| > |g(z)| \ \forall z \in C$

$\Rightarrow f, f+g$ same # zeros in C

Proof Idea:

$f_t(z) = f(z) + t g(z)$ $\quad n_t =$ # of zeros of f_t in C

Arg Princ $\Rightarrow n_t = \underbrace{\int \frac{f_t'}{f_t}}$ continuous in t and $n_t \in \mathbb{Z}$ so n_t constant.

Open Mapping Theorem:

f holomorphic + nonconstant

$\Rightarrow f$ open map

Proof Idea: $f(z_0) = w_0$

$g(z) = \underbrace{f(z) - w_0}_{F(z)} + \underbrace{w - w_0}_{G(z)}$ $\quad$ circle $|z - z_0| = \delta$ with $\varepsilon \le |f(z) - w_0|$

apply Rouche's to $F, G \Rightarrow F$ root $\rightsquigarrow g$ root $\rightsquigarrow w \in \mathrm{Im}(f)$.

Max Modulus Principle:

f holomorphic + nonconstant

$\Rightarrow f$ has no max in open Ω

Proof Idea:

open mapping $\Rightarrow f$ open

If $f(z_0)$ is max, take $z_0 \in \mathcal{U}$

then $|f(z_0)|$ not max in $|f(\mathcal{U})|$ contradiction.

Holomorphic Functions

Defns

complex differentiable

- f is holomorphic at $z_0 \in \Omega$ if
  $$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \quad \left(\begin{array}{l} h \to 0 \text{ in any} \\ \text{way in } \mathbb{C} \end{array} \right)$$
  and f is holomorphic on Ω if it is $\forall z \in \Omega$.
- If f is holomorphic on $\mathbb{C}$ it is entire.

If Ω is closed then holo. on open containing Ω

Examples

- polynomials (same f' as usual)
- $1/z$ on Ω if $0 \notin \Omega$ $\quad (f' = -1/z^2)$
- Non-example: $f = \bar{z}$ $\quad \frac{f(z_0+h) - f(z_0)}{h} = \frac{\bar{h}}{h}$ $\begin{array}{l} \nearrow 1 \ (h = r) \\ \text{no lim} \\ \searrow -1 \ (h = ir) \end{array}$
- power series with radius of convergence
  - $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\qquad R = \infty \quad$ (hol. on $\mathbb{C}$)
  - $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $\qquad R = \infty \quad$ (hol. on $\mathbb{C}$)
  - $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ $\qquad R = \infty \quad$ (hol. on $\mathbb{C}$)

Properties

f, g holomorphic in Ω

- $f+g$, fg holomorphic in Ω w/ usual derivatives
- $g(z_0) \neq 0$ then f/g hol at z_0 w/ usual derivative
- chain rule holds $(g(f(z)))' = g'(f(z)) f'(z) \ \forall z \in \Omega$.
- f holomorphic $\iff f(z+h) - f(z) = ah + h\psi(h)$ where $a = f'(z)$ and $\psi(h) \to 0$ as $h \to 0$.

Cauchy - Riemann Equations

Thm:
If we write $z = x+iy$ and $f(x,y) = u(x,y) + i\,v(x,y)$

f holomorphic $\Longrightarrow$ $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Pf:
Take $h = r \in \mathbb{R}$ $r \to 0$ for limit

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

Take $h = ir, r \in \mathbb{R}$ $r \to 0$ for limit

$f'(z) = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Now equate real & imaginary parts.

Thm (converse)
$f(z) = u(z) + i\,v(z)$
u, v continuously diff.
+ satisfy Cauchy-Riemann Eqs
in Ω ~~[illegible]~~
$\Longrightarrow$ f is holomorphic
$f' = 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

Power Series

Defns

- The (complex) series $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ <u>converges absolutely</u> if the (real) series $\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n| \, |z|^n$ converges.
- Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there is some <u>radius of convergence</u> $0 \leq R \leq \infty$ s.t.
  - $|z| < R \longrightarrow$ the series converges absolutely
  - $|z| > R \longrightarrow$ the series diverges

  and the region $|z| < R$ is the <u>disc of convergence</u>.
- <u>Hadamard's Formula</u> $1/R = \limsup |a_n|^{1/n}$ (where $1/0 = \infty$ and $1/\infty = 0$)

Examples

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad R = \infty$
- $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ $\quad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

<u>Euler Formulas:</u>

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{iz} = \cos(z) + i\sin(z).$$

Taylor Series

Defn

- f is <u>analytic at z_0</u> if $\underbrace{f(z) = \sum a_n(z-z_0)^n}_{\text{has positive radius of convergence}}$ in some neighborhood of z_0

  and f is <u>analytic on Ω</u> if it is $\forall z \in \Omega$

- The <u>Taylor series expansion</u> of f at z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Results

- power series $\Rightarrow$ holomorphic, infinitely diff (same radius!)

  Let $f(z) = \sum a_n z^n$ then $f'(z) = \sum n a_n z^n$

  and this has the same radius of convergence

  by Hadamard's Formula $\limsup |a_n|^{1/n} = \limsup |na_n|^{1/n}$.

- analytic $\Rightarrow$ holomorphic

  $\downarrow$

  gives a power series which is holomophic where $|z| < R$.

- holomorphic $\Rightarrow$ analytic

  pf by Cauchy Integral formula.

Complex Integration

Defns:

- $z(t) : [a,b] \to \mathbb{C}$ is a <u>parametrized curve</u>
- If $z'(t)$ exists and is continuous it is <u>smooth</u>
- If $z(a) = z(b)$ it is <u>closed</u>
- If z is injective (curve not self intersecting) it's <u>simple</u>
- Given γ and a parametrization $z : [a,b] \to \mathbb{C}$

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt} \leftarrow \text{path integral}$$

This is independent of parameterization

- $\text{length}(\gamma) := \int_a^b |z'(t)|\,dt$

Examples:

$z(t) = e^{it}$ $t \in [0, 2\pi]$

unit circle

$f(z) = \frac{1}{z}$ $\quad \int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$

$f(z) = z^2$ $\quad \int_\gamma f(z)\,dz = \int_0^{2\pi} e^{2it}\, i e^{it}\,dt = i\left[\frac{1}{3i} e^{3it}\Big|_0^{2\pi}\right] = 0$

Properties:

- $\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz$ $\quad \gamma^-$ has reverse orientation
- $\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma)$

Primitives

Defn A <u>primitive</u> of f on Ω is some F s.t.

- F is holomorphic on Ω
- $F'(z) = f(z)$ for all $z \in \Omega$

Thm If F is a primitive of f on Ω and $\gamma \subset \Omega$ starts at w_1 and ends at w_2

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1)$$

Pf: $\int_\gamma f\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt = \int_a^b \frac{d}{dt}F(z(t))\,dt = F(z(b)) - F(z(a))$

Cor: If γ is closed, f has primitive then $\int_\gamma f(z)\,dz = 0$.

Pf: $\int_\gamma f(z)\,dz = F(w_2) - F(w_2) = 0$.

Cor: f holomorphic on connected Ω with $f' = 0 \longrightarrow f$ is <u>constant</u>.

Pf: f a primitive for f'. $\gamma_w : w_0 \longrightarrow w$ (fixed w_0).

$0 = \int_{\gamma_w} f'(z)\,dz = f(w) - f(w_0)$ so $f(w) = f(w_0)$ $\forall w \in \Omega$.

Cauchy's Theorem

Cauchy's Theorem If f is holomorphic in a region Ω (or disc) and γ is smooth closed curve then

$$\boxed{\int_\gamma f(z)\,dz = 0}$$

Pf: (via Green's Thm)

$f = u + iv$ $\quad dz = dx + i\,dy$ (can make rigorous by param. $\gamma(t)$)

$$\int_\gamma f(z)\,dz = \int_\gamma (u(z)+iv(z))(dx+i\,dy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

Green's Thm:

L, M cts partial deriv. C curve, D region in curve

$$\int_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dx\,dy$$

so $L = u$ $\quad M = -v$ $\qquad \int_\gamma (u\,dx - v\,dy) = -\iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy = 0$

and $L = v$ $\quad M = u$ $\qquad \int_\gamma v\,dx + u\,dy = \iint \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\text{Cauchy Riemann Eqs} \Rightarrow\, =0} dx\,dy = 0$

Alt Pf (Goursat's)

Applies if u, v are just differtiable (not nec. cont. diff)

Goursat's Thm: f diff (holom) T triangle $\int_T f(z)\,dz = 0$.

Construct a primitive $F(z) = \int_{\gamma_z} f(w)\,dw$ $\quad \gamma_z : z_0 \longrightarrow z$.

use Goursat's and cancel edges to show F is diff with $F' = f$ so that f has primitive and $\int_\gamma f(z)\,dz = 0$.

Cauchy's Integral formula

Cauchy Integral Formula:

f holomorphic in open disc D and its boundary C $\Big\}$ $\boxed{f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw}$

Pf:

$F(w) = \frac{f(w)}{w-z}$ holo. on  keyhole contour,

let consider width $\to 0$ to get big circle C, little C_ε.

$0 = \int_{\text{contour}} F(w)dw = \int_C F(w)dw + \int_{C_\varepsilon} F(w)dw$ (C, C_ε opposite orientations)

compute inner circle integral

$F(w) = \frac{f(w)}{w-z} = \underbrace{\frac{f(w)-f(z)}{w-z}}_{\substack{\text{as } w\to z \text{ (i.e. } \varepsilon\to 0) \\ \text{goes to } f'(z) \text{ so bdd.}}} + \frac{f(z)}{w-z}$ $\to \int_{C_\varepsilon} \frac{f(w)-f(z)}{w-z}dw \to 0.$

$\int_{C_\varepsilon} F(w) = \int_{C_\varepsilon} \frac{f(z)}{w-z}dw = f(z)\int_0^{2\pi} \frac{1}{\varepsilon e^{-it}} \, -i\varepsilon e^{-it} dt = -f(z)2\pi i.$

$\int_C F(w)dw = -\int_{C_\varepsilon} F(w)dw = f(z)2\pi i \Rightarrow f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw.$

Morera's Theorem

converse to Cauchy's Theorem

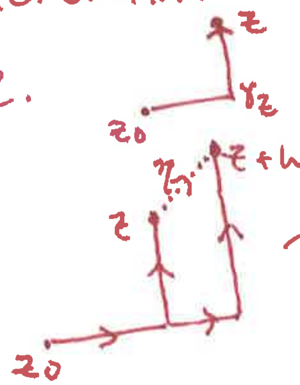
Morera's Thm –

f continuous on open disc D, $\forall$ triangles $T \subset D$ $\int_T f(z)\,dz = 0$ $\Big\}$ f is holomorphic.

Pf:

Goal: construct antiderivative F show it is holomorphic.
holomorphic $\Rightarrow$ infinitely differentiable so $f = F'$ is too.

Construction:
Fix $z_0 \in D$. Define $\gamma_z : z_0 \to z$.
Define $F(z) = \int_{\gamma_z} f(w)\,dw$.



so well-defined

$$F(z+h) - F(z) = \int_{z+h} f\,dw - \int_z f\,dw$$
$$= \int_\eta f(w)\,dw$$
(by continuity)
$$= \int_\eta f(z) + \psi(w)\,dw = hf(z) + \int_\eta \psi(w)\,dw \to hf(z)$$
as $h \to 0$.

so $F'(z) = f(z)$, F is holomorphic $\rightsquigarrow$ f is holomorphic.

Cauchy's Int. Form. Higher Derivatives

CIF Higher Derivatives:

f holomorphic in open Ω and C circle in Ω $\Big\}$ $\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw}$

<u>Pf</u>: Induction on n.

<u>$n=0$:</u> Cauchy Integral Formula

<u>$n>0$:</u>

Assume $f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^n}\,dw.$

$$f^{(n)}(z) = \lim_{h\to 0}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \lim_{h\to 0}\frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{\boxed{h}}\underbrace{\left[\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}\right]}\,dw$$

$$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \dots + AB^{n-2} + B^{n-1})$$

$$\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n} = \underbrace{\left(\frac{1}{w-z-h} - \frac{1}{w-z}\right)}_{=\frac{\boxed{h}\text{ cancels}}{(w-z-h)(w-z)}}\underbrace{\sum\left(\frac{1}{w-z-h}\right)^k\left(\frac{1}{w-z}\right)^{n-1-k}}_{\to\, n\cdot\frac{1}{(w-z)^{n-1}}}$$

$$f^{(n)}(z) = \cancel{\lim_{h\to 0}}\frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,n\,dw = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw. \quad \square$$

Cauchy's Inequalities

Cauchy's Inequalities:

f holomorphic in open set containing closure of a disc D w/ center z_0 and R radius and boundary C

$$\boxed{\begin{gathered} |f^{(n)}(z_0)| \leq \frac{n!\,\|f\|_C}{R^n} \\ \|f\|_C = \sup_{z \in C} |f(z)| \end{gathered}}$$

Pf:

By Cauchy Int Form. for higher derivatives

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw$$

parametrization of C by θ

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}}\, iRe^{i\theta}\, d\theta \right|$$

$$= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n}}\, d\theta \right|$$

$$\leq \frac{n!}{2\pi} \frac{1}{R^n} \|f\|_C\, 2\pi = \frac{n!\,\|f\|_C}{R^n} \qquad \square$$

Liouville's Theorem

Liouvilles Thm:

f entire + bounded $\Longrightarrow$ f constant

Pf:

Goal: show $f' = 0$

f bdd $\rightsquigarrow$ $|f| \le B$ everywhere

Cauchy Integral Formulas $\to$ Cauchy Inequalities so

$$|f'(z_0)| \le \frac{\|f\|_C}{R} \le \frac{B}{R} \to 0 \text{ as } R \to \infty$$

so $f'(z_0) = 0$ $\forall z_0 \in \mathbb{C}$.

Then f is primitive of f' to get f constant.

Modification #1

f entire, $f^{(n)}$ bounded

$\Longrightarrow$ f polynomial deg $= n$

Pf:

f entire $\Rightarrow$ $f^{(n)}$ entire

+ bdd $\Rightarrow$ $f^{(n)}$ constant.

use anti-derivatives and power series exp to get f polynomial of degree n

Modification #2

f entire, $\operatorname{Im}(f)$ bounded

$\Longrightarrow$ f is constant

Pf:

Take $F(z) = e^{-if(z)}$

$F(z)$ entire by composition

$f(z) = u(z) + iv(z)$ ← bdd

$$F(z) = e^{-iu(z) + v(z)} = \underbrace{e^{-iu(z)}}_{\text{bdd by } 1} \; \underbrace{e^{+v(z)}}_{\text{bdd b/c } v(z) \text{ is}}$$

so $F(z)$ constant $\Rightarrow f$ constant

Fundamental Theorem of Algebra

FTA Statement:

$P(z)$ nonconstant polynomial in $\mathbb{C}[x]$ $\Bigg\}$ $P(z)$ has a root in $\mathbb{C}$ ($\Leftrightarrow$ splits completely in $\mathbb{C}$)

Pf:

Assume $P(z)$ has no roots & show $P(z)$ constant.

Take $1/P(z)$ which has no poles $\longrightarrow$ entire.

Sufficient to show bounded.

$|P(z)| \longrightarrow \infty$ as $|z| \longrightarrow \infty$ so for $R >> 0$

if $|z| > R$ $\quad |P(z)| > \frac{1}{M} \rightsquigarrow |1/P(z)| < M$.

Now for $|z| \leq R$ we have a closed, finite region.

If unbounded, $|1/P(z)| \to \infty \longrightarrow |P(z)| \to 0$

and continuity implies $P(z)$ has a root, contradiction.

$\to 1/P(z)$ bdd + entire $\Rightarrow 1/P(z)$ constant

$\Rightarrow P(z)$ constant $\lightning$

So $P(z)$ has a root in $\mathbb{C}$.

Analytic = Holomorphic

Defns

- f is holomorphic on Ω if $f'(z)$ exists $\forall z \in \Omega$
- f is analytic on Ω if $f(z) = \underbrace{\sum a_n (z-z_0)^n}_{\text{positive radius of conv.}}$ in a nbhd of z_0, $\forall z_0 \in \Omega$

Holomorphic $\Rightarrow$ Analytic

take $z_0 \in \Omega$ and take open disc D centered @ z_0 bdry C.

Cauchy Integral Formula

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$

(introduce $z-z_0$ & get expansion)

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\,\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} = \underbrace{\frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw$$

switch int. & sum by uniform conv.

$$= \sum_{n=0}^{\infty}\left(\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)\frac{(z-z_0)^n}{n!}$$

apply CIF for higher derivatives

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$w \in C$ $z \in D$ z_0 at center so $\left|\frac{z-z_0}{w-z_0}\right| < r < 1$

$\rightarrow$ positive radius of convergence.

Analytic $\Rightarrow$ Holomorphic

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\sum_{n=0}^{\infty}\frac{a_n}{h}\left[(z+h-z_0)^n-(z-z_0)^n\right] \overset{\text{uniform conv.}}{=} \sum_{n=0}^{\infty} a_n n (z-z_0)^{n-1}$$

Hadamard's Formula says same radius of convergence.

Poles and Residues

Defns

- z_0 is a zero when $f(z_0)=0$, and $f(z)=(z-z_0)^n g(z)$ where g nonvanishing in nbhd of z_0, z has multiplicity n
- f defined in deleted nbhd of z_0 $\{0<|z-z_0|<r\}$, and $(1/f)(z_0)=0$ gives a holomorphic function $1/f$, then f has pole at z_0 and $f(z)=(z-z_0)^{-n}h(z)$ gives a multiplicity / order of n.
- simple poles have order 1.
- $$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-1}}{(z-z_0)}}_{\text{principal part of } f} + G(z)$$

  residue of f at z_0

  the residue at z_0 is $\text{res}_{z_0}(f) = a_{-1}$.

Residues as limits:

z_0 simple pole $\Longrightarrow$ $\text{res}_{z_0}(f) = \lim_{z\to z_0} (z-z_0)f(z)$

z_0 pole, order n $\Longrightarrow$ $\text{res}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z)$

Residues via Power Series:

$$\text{Res}_0\left(\frac{e^z}{z^3}\right) = \text{Res}_0\left(\frac{1}{z^3}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)\right) = \text{Res}_0\left(\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2z}+\dots\right) = 1/2$$

Residue Formula

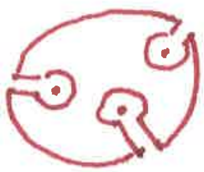
Theorem:

f holomorphic in open set containing circle C & interior **except** for poles at $z_1, z_2, \ldots, z_N$ (inside C)

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k}(f)}$$

Pf:

Take multiple keyhole contour



send corridor widths to 0 $\Longrightarrow \displaystyle\int_C f(z)\,dz = \sum_{k=1}^{N} \int_{C_k} f(z)\,dz$

For a pole z_0 and mini circle C_ε, expand $f(z)$ in nbhd

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2}} + \underbrace{\frac{a_{-1}}{(z-z_0)}} + \underbrace{G(z)}$$

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{a_{-k}}{(z-z_0)^k}\,dz = g_k^{(k-1)}(z) \overset{k>1}{=} 0$$

where $g_k(z) = a_{-k}$ (constant)

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{a_{-1}}{(z-z_0)}\,dz = g_1(z) = a_{-1}$$

holomorphic so Cauchy's Thm $\displaystyle\frac{1}{2\pi i}\int_C G(z)\,dz = 0$

so $\displaystyle\frac{1}{2\pi i}\int_{C_\varepsilon} f(z)\,dz = a_{-1} = \operatorname{res}_{z_0}(f)$ and summing over all poles and moving over $2\pi i$ gives desired result. $\square$

Residue Theorem Computations

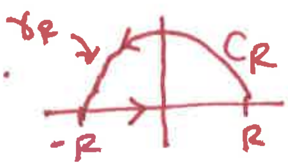
Steps:

Goal: $I := \int_a^b f(x)\,dx$

1. Choose $g(z)$ with $g(z) = f(x)$ on $\mathbb{R}$ or $f(x) = \mathrm{Re}(g(z))$.
2. Pick contour C including axis for I and computable parts.
3. Compute $\int_C g(z)\,dz$ using Residue Thm or parametrization
4. Break up $\int_C g(z)\,dz$ and compute other parts (not I)
5. Solve for I

Example:

Compute $\int_0^\infty \frac{1}{1+x^2}\,dx$.

1. $g(z) = \frac{1}{1+z^2}$ has poles at $\pm i$
2. 

contour C_R contains simple pole i.
3. $\int_{C_R} g(z)\,dz = 2\pi i \, \mathrm{Res}_i\, g(z) = 2\pi i \lim_{z \to i} (z-i)\frac{1}{(z+i)(z-i)} = \frac{2\pi i}{i+i} = \pi$
4. $\int_{C_R} g(z)\,dz = \int_{-\infty}^{\infty} g(z)\,dz + \int_{\gamma_R} g(z)\,dz = 2\int_0^\infty \frac{1}{1+x^2}\,dx + \int_0^\pi \frac{1}{1+R^2 e^{2it}} iRe^{it}\,dt$

   $\leq \pi \left\| \frac{iRe^{it}}{1+R^2e^{2it}} \right\| \longrightarrow 0$ as $R \to \infty$
5. $\pi = \int_{C_R} g(z)\,dz = 2\int_0^\infty \frac{1}{1+x^2}\,dx \implies \int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$

Meromorphic

Defns:

- a function is meromorphic in Ω if it is holomorphic on $\Omega - \underbrace{\{z_1, z_2, \dots\}}_{\text{isolated points (no limit in } \Omega)}$ w/ poles @ $z_1, z_2, \dots$
- $f(z)$ has a pole at infinity if $F(z) = f(1/z)$ has a pole at 0.
- $f(z)$ is meromorphic on the extended plane if meromorphic on $\mathbb{C}$ and $F(z) = f(1/z)$ is either holomorphic or has pole at 0.

Thm:

meromorphic functions on the extended plane are exactly rational functions $\frac{p(z)}{q(z)}$ ← polynomials.

Laurent Series

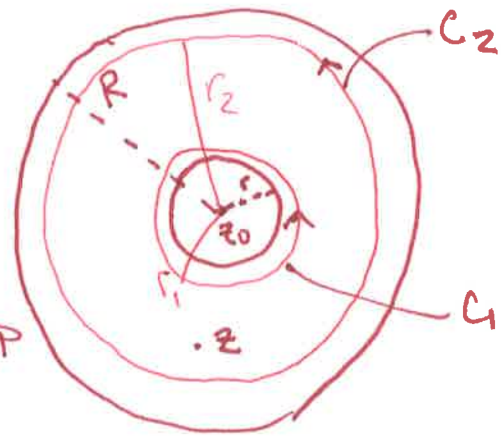
Defn

· Laurent Series ∈ $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ (infinite neg. extension of Taylor series)

Laurent's Thm (Existence)

f holomorphic on $r<|z-z_0|<R$ then $\forall z$ $r<|z-z_0|<R$

$$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$



Pf: choose $r<r_1<|z-z_0|<r_2<R$

$C_1: |z-z_0|=r_1$ $\quad$ $C_2: |z-z_0|=r_2$

Taking keyhole of C_1, C_2 & Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw-\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw$$

expand $\frac{1}{w-z}=\frac{1/w-z_0}{1-\left(\frac{z-z_0}{w-z_0}\right)}=\frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ $\quad$ $\left|\frac{z-z_0}{w-z_0}\right|<1$ since $w\in C_2$ and z inside.

expand $\frac{1}{w-z}=\frac{1/z-z_0}{1-\left(\frac{w-z_0}{z-z_0}\right)}=\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n$ $\quad$ $\left|\frac{w-z_0}{z-z_0}\right|<1$ since $w\in C_1$.

put together to get expansion w/ $a_n=\frac{1}{2\pi i}\int_{C_n}\frac{f(w)}{(w-z_0)^{n+1}}\,dw$.

Uniqueness (similar to Residue Thm)

To show $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n=\sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ means $a_n=b_n\ \forall n$.

Take $(z-z_0)^{-k-1}$ multiply, integrate. If $n=k$, $\int a_k(z-z_0)^{-1}=2\pi i a_k$ and all others are 0, so $a_k=b_k$. Can do for any k so is unique.

Argument Principle

Thm:

f meromorphic in Ω containing circle C & its interior

If f has no poles/zeros on C then

$$\boxed{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = (\#\text{ of zeros of } f \text{ in } C \text{ w/ mult.}) - (\#\text{ of poles of } f \text{ in } C \text{ w/ mult.})}$$

Pf:

Determine poles & residues of f'/f and apply Residue Thm

- zero of f order n $\Rightarrow f(z) = (z-z_0)^n h(z) \Longrightarrow \dfrac{f'(z)}{f(z)} = \dfrac{n}{z-z_0} + \dfrac{h'(z)}{h(z)}$ simple pole res $= n$
- pole of f order n $\Rightarrow f(z) = (z-z_0)^{-n} h(z) \Rightarrow \dfrac{f'(z)}{f(z)} = \dfrac{-n}{z-z_0} + \dfrac{h'(z)}{h(z)}$ simple pole res $= -n$

Residue Thm:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum \mathrm{Res}_{z_i}\left(\frac{f'}{f}\right) = (\#\text{ of zeros of } f \text{ w/ mult.}) - (\#\text{ of poles of } f \text{ w/ mult.}) \quad \square$$

Idea:

$\dfrac{f'(z)}{f(z)} =$ derivative of $\log(f(z))$ $= \log|f(z)| + i\underbrace{\arg f(z)}_{\text{determined up to } 2\pi k}$

$\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\,dz =$ change in argument along curve γ

Rouche's Theorem

consequence of Argument Principle

Theorem:
f, g holomorphic on Ω containing circle C & its interior

$|f(z)| > |g(z)|$ $\forall z \in C \Rightarrow f, f+g$ have same # of zeros inside C.

Pf: $t \in [0,1]$
$f_t = f(z) + tg(z)$ $\quad f_0 = f$, $f_1 = f+g$ $\quad n_t =$ # of zeros of f_t inside C $\in \mathbb{Z}_{\geq 0}$

since $|f| > |g|$, $f_t \neq 0$ on C so argument principle

$$n_t - \underset{\substack{\uparrow \\ \text{holo} \Rightarrow \text{no poles}}}{0} = \underbrace{\frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}\,dz}_{\text{show that this is cts in } t}$$

$f_t'(z), f_t(z)$ joint cts in z, t and $f_t(z) \neq 0$ on C
so $f_t'(z)/f_t(z)$ also joint cts in $z, t \to \int \frac{f_t'}{f}dt$ cts in t.

cts integer valued functions are constant $\Rightarrow n_0 = n_1$ $\square$

Open Mapping Theorem

Thm:

f holomorphic + nonconstant in a region Ω $\Big\}$ f is an open map ($\hookrightarrow$ maps opens to opens)

Pf:

$w_0 = f(z_0)$ in image, want some nbhd $|w - w_0| < \varepsilon$ s.t. $w = f(z)$ for some z.

Define

$$g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$$

WTS $g(z)$ has a zero when $|w - w_0| < \varepsilon$ for choice of ε.

choose $\delta > 0$ s.t. $\{|z - z_0| \leq \delta\} \subset \Omega$

on $\{|z - z_0| = \delta\}$ $f(z) \neq w_0$

since f holo & nonconstant.

and $\varepsilon > 0$ so on $\{|z - z_0| = \delta\}$ we have $|f(z) - w_0| \geq \varepsilon$.

on the circle $|z - z_0| = \delta$

$|f(z) - w_0| = |F(z)| \geq \varepsilon > |w - w_0| = |G(z)|$

so by Rouche's $F(z)$ and $F(z) + G(z) = g(z)$ have same # of roots in $|z - z_0| < \delta$.

$F(z)$ has root at z_0 so $g(z)$ has a root

which implies $w \in \operatorname{Im}(f)$ as desired.

Maximum Modulus Principle

Thm:

f nonconstant, holomorphic in region Ω $\Rightarrow$ f cannot attain a maximum in Ω.

Pf:

By open mapping theorem.

Suppose f has max at $z_0 \in \Omega$ (open)

so $|f(z_0)| \geq |f(z)|$ $\forall z \in \Omega$.

f nonconstant + holo $\Longrightarrow$ f open mapping

choose $z_0 \in \mathcal{U} \subset \Omega$ then $f(\mathcal{U})$ open and contains $f(z_0)$ so by topology there is some $z \in \mathcal{U}$ s.t. $|f(z)| > |f(z_0)|$, a contradiction. So no max in Ω. $\square$

~~Note~~ Cor:

on region Ω w/ compact closure $\bar{\Omega}$,

f cts on $\bar{\Omega}$ & holom. on Ω $\quad \sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \bar{\Omega} - \Omega} |f(z)|$

maximum occurs on the boundary $\rightarrow$

Complex Logarithm

$z = re^{i\theta}$

want $\log(z) = \log r + i\theta$] not well defined.

only defined up to $2\pi n$

Restricting to "local" setting where θ cannot loop defines a <u>branch</u> of the logarithm.

<u>Thm</u>

Ω simply connected, $1 \in \Omega$, $0 \notin \Omega$ } $\exists$ a branch of $\log_\Omega(z) = F(z)$

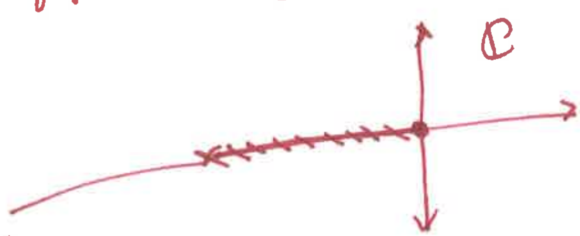
(i) F holomorphic in Ω

(ii) $e^{F(z)} = z \quad \forall z \in \Omega$

(iii) $F(r) = \log(r)$ near 1.

<u>Pf Idea</u>:

Define $F(z) = \int_\gamma f(w)\,dw \qquad \gamma: 1 \longrightarrow z$.



<u>Example</u>:

split plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$

principal branch $\log(z) = \log(r) + i\theta \qquad |\theta| < \pi$